COLLISION OF PLANE JETS OF PSEUDOPLASTIC LIQUID WITH A FREE BOUNDARY

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Experimental research on the phenomenon of plane cumulation shows that the theory of the process, based on the model of an ideal liquid of [1, 2], needs improvement. The model of a Newtonian liquid used in [3], while qualitatively explaining certain experimental relationships, encounters difficulties in describing the process as a whole [4]. These difficulties are connected, on the one hand, with the fact that the velocity of the cumulative jet and the shape of the marker line in the plates calculated from the model of a viscous liquid (with a constant viscosity) in one and the same test coincide with the respective experimental characteristics for considerably different Reynolds numbers: In a cumulative jet coincidence is reached at Re ≈ 350 and in colliding plates at Re ≈ 25. Thus, the effective viscosity in a jet and in plates proves to be different, contradicting the initial assumption that the viscosity is constant. On the other hand, in a given regime of collision (the velocity, thickness, and density of the plates are given) the viscosity cannot be determined before the test: It is determined in the tests by matching the calculated and experimental characteristics and, generally speaking, it varies as a function of the experimental conditions. Consequently, the effective viscosity is not an objective characteristic of a material. The option of such an objective parameter would be very desirable. More complicated rheological laws must be used to overcome the difficulties.

In the present article it is proposed to describe the actual process on the basis of a model of a power-law pseudoplastic liquid, the parameters of which are chosen from independent measurements of the dynamic plasticity of metals.

1. Choice of the Rheological Relation $\sigma = \sigma(\dot{\varepsilon}, T)$

As was shown in [4], the effective Reynolds numbers in a cumulative jet (Re^0) and in a ramming jet (Re_*) differ by about an order of magnitude: $Re^0 \approx 350$ and $Re_* \approx 25$. Therefore, it is natural to try to find another characteristic which varies insignificantly in one and the same process. For this purpose we estimate the characteristic shear stresses in a jet and a ram from the relation $\sigma = 2\mu\dot{\varepsilon}$.

For the ratio σ^0/σ_*, where a superscript corresponds to a jet while a subscript corresponds to a ram, we have

$$\sigma^0/\sigma_* = \mu^0\dot{\varepsilon}^0/(\mu_*\dot{\varepsilon}_*) = Re_*\dot{\varepsilon}^0/(Re^0\dot{\varepsilon}_*). \tag{1.1}$$

We estimate the characteristic shear velocities $\dot{\varepsilon}^0$ and $\dot{\varepsilon}_*$ in a jet and a ram as the ratio of the difference between the velocities at the free boundary ($v \approx U$) and at the critical point ($v = 0$) to the minimum distance r from the critical point to the given free boundary [$r^0 \approx h\cdot(1 - \cos\gamma)/2$, $r_* \approx h(1 + \cos\gamma)/2$],

$$\dot{\varepsilon}^0 \approx 2U/[h(1-\cos\gamma)],\ \dot{\varepsilon}_* \approx 2U/[h(1+\cos\gamma)], \tag{1.2}$$

where U is the velocity of inflow of material in the frame of reference of the contact point; h is the thickness of the inflowing jets; γ is half the collision angle. The case of U = 1.0 km/sec, h = 4 mm, and 2γ = 45° was investigated in [4]. Substituting these values into (1.2), we obtain

$$\dot{\varepsilon}^0/\dot{\varepsilon}_* = (1+\cos\gamma)/(1-\cos\gamma) \approx 24.5. \tag{1.3}$$

The substitution of (1.3) into (1.1) results in $\sigma^0/\sigma_* \approx 1.75$.

Thus, compared with the scale of variation of the viscosity, the characteristic shear stresses are approximately constant in the region of high pressures.

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This conclusion is not surprising from the point of view of the theory of plasticity. In processes of pressing, stamping, and drawing of metals, when the deformation rate is not too high, the experimental Tresca law, according to which, in a state of fluidity, the highest shear stress is constant at all points of the medium and equal to the yield point of the material in pure shear. In later research [5-7] it was established that the behavior of metals is better described by the dependence

$$\sigma_i = \sigma_y, \tag{1.4}$$

called the Mises plasticity condition. In (1.4), σ_i is the stress intensity and σ_y is the yield point in uniaxial stretching.

At $\dot{\varepsilon} \approx 10^5$ sec^{-1} and higher the law of deformation of metals can differ from (1.4). There are no systematic experimental data in this range. At deformation rates $\dot{\varepsilon} \approx 10^4$ sec^{-1} and lower, however, information about the behavior of metals is quite detailed [8-13]. It was established that as $\dot{\varepsilon}$ increases the breaking point σ_b (the yield point σ_y) increases. And the dependences of σ_y on $\dot{\varepsilon}$ and T split into two characteristic sections,

$$\sigma_T = \sigma_T^0 e^{D/T} \dot{\varepsilon}^n; \tag{1.5}$$

$$\sigma_y = \sigma_y^1 \dot{\varepsilon}^{m(T-T_0)}, \tag{1.6}$$

where $\sigma^0{}_y$, D, N, $\sigma^1{}_y$, m, and T_0 are certain constants of the material. The constant n equals 0.018, 0.020, 0.030, and 0.025 for copper, aluminum, 0.2% C steel, and lead, respectively, while the product $m(T - T_0)$ does not exceed 0.20 in the entire temperature range investigated (up to 1200°C for steel and up to the melting temperatures for the other metals). The function (1.5) is valid for $T < T_0$, where T_0 = 370, 340, 430, and 80°K for copper, aluminum, 0.2% C steel, and lead, respectively. For $T > T_0$ there exists a critical deformation rate $\dot{\varepsilon}_*(T)$ such that (1.5) is satisfied for $\dot{\varepsilon} > \dot{\varepsilon}_*$ while (1.6) is satisfied for $\dot{\varepsilon} < \dot{\varepsilon}_*$. These transitions from the function (1.5) to (1.6) were discovered in [8, 9, 14] for lead, aluminum, and zinc. In [9] the hypothesis was expressed that similar behavior during deformation is valid for other metals. If one holds to this assumption, then in extrapolating the known experimental data into the region of $\dot{\varepsilon} \gtrsim 10^4$ sec^{-1} one must choose the function (1.5) as the most probable. In a graphic extrapolation of data on copper presented in [8, 9] a plausible value of D is D = 150°K. Since the value of D is introduced, the value of $\sigma^0{}_y$ can be determined by requiring that for T = 300°K and $\dot{\varepsilon}$ = 1 sec^{-1} (the usual conditions for tests of metals) the value of σ_y coincides with the reference value of the yield limit in uniaxial stretching. In future we shall understand $\dot{\varepsilon}$ as the intensity of the deformation rate.

We introduce the equation of state of metals under the conditions of high-speed deformation in the form

$$\sigma_{ij} = \sigma_0\delta_{ij} + A(T)J^{\alpha}\dot{\varepsilon}_{ij}, \quad i, j = 1, 2, 3, \; \dot{\varepsilon}_{ii} = 0, \tag{1.7}$$

where σ_{ij} and $\dot{\varepsilon}_{ij}$ are the stress and deformation-rate tensors; J is the intensity of the deformation rate, $J = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}}$; $\sigma_0 = \frac{1}{3}\sigma_{ii}$; $A(T) = \frac{2}{3}\sigma_y^0 e^{D/T}$; $\alpha = -1 + n$.

It is simple to trace the connection between (1.5) and (1.7). Under the conditions of high-speed jet flow of metals the maximum pressures usually do not exceed 35 GPa, whereas the bulk modulus is K ≈ 200 GPa, i.e., the compressibility of the metal can be neglected ($\dot{\varepsilon}_{ii}$ = 0). Further, the shear deformation under the conditions of jet flow is ~10-100%, so that we shall neglect the initial elastic section (less than 1%) in the deformation of the metal. In [15] it is pointed out that at high pressures metals deform by hundreds of percent without destruction, and hence one can assume that during the entire process of deformation a metal is in a fluid state and, in accordance with the theory of plasticity, $\sigma_i = \sigma_y$ [Eq. (1.4)]. Since $\sigma_y = \sigma_y(J, T)$, substituting (1.5) into (1.4), we obtain

$$\sigma_i = \sigma_y^0 e^{D/T} J^n. \tag{1.8}$$

The condition (1.8) means that, in the space of principal stresses, the trace of the fluidity surface on the deviator plane is a circle, the radius of which depends on the temperature T and the intensity J. The case when the radius of the circle is a function of the parameter q characterizing the strength has come to be analyzed in the theory of flow [16].

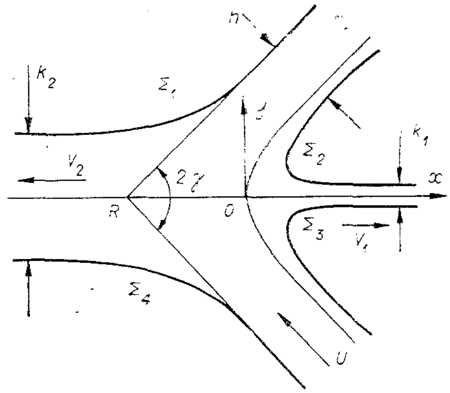

Fig. 1

Formally, (1.8) does not differ from this case, since the origin of the parameter q is not important for the derivation of the final equations. Therefore, one can write at once the Saint Venant–Levi–Mises flow equations,

$$\dot{\varepsilon}_{ij} = \frac{3}{2}\frac{J}{\sigma_i} s_{ij}, \; i, j = 1, 2, 3, \tag{1.9}$$

where $\sigma_i = \sqrt{\frac{3}{2} s_{ij} s_{ij}}$, $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_0$. Substituting (1.8) into (1.9), we obtain (1.7).

In the plane case (1.9) is written in the form

$$\dot{\varepsilon}_{ij} = \frac{\sqrt{\dot{\varepsilon}_{11}^2 + \dot{\varepsilon}_{12}^2}}{\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}} s_{ij}, \; i, j = 1, 2, \; \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} = \frac{\sigma_y}{\sqrt{3}},$$

while the constants A and σ_0 in (1.7) will equal

$$\sigma_0 = \frac{\sigma_{11} + \sigma_{22}}{2}, \quad A(T) = \frac{\sigma_y^0}{\sqrt{3}} e^{\frac{D}{T}}.$$

For T = const a medium of the type (1.7) is known in the literature as a generalized Newtonian (power-law, Ostwald–de Waele) liquid; for $\alpha > 0$ the liquid is called dilatant, for $-1 < \alpha < 0$ it is pseudoplastic, and the case of $\alpha = 0$ corresponds to a Newtonian liquid. It is seen from (1.7) that all metals can be considered as pseudoplastic liquids.

As an analysis of the literature shows, after a Newtonian liquid, a power-law liquid is among the most studied. The flows of many polymers, solutions, and suspensions obey a power law. An exact solution on flow in pipes and approximate solutions for flows around a sphere [17, 18], flows in boundary layers near a solid wall [19, 20], and flows with mixing are known for this liquid. Judging from the known literature sources, boundary layers near a free surface of a power-law liquid and the applicability of the model of a power-law pseudoplastic liquid to high-speed metal flows have not been analyzed.

Estimates of the temperature in the flow region show that the heating in a ram is not very high (the temperature rise is about 50-80°C), while the temperature in a cumulative jet should not exceed 600-700°C. In such a temperature range (300-1000°K) the variation of $\sigma_y(\dot{\varepsilon}, T)$ proves to be small for $\dot{\varepsilon}$ = const. But the variation of the deformation rate in the flow region is great ($\dot{\varepsilon}$ is small in the incoming jets and $\dot{\varepsilon} \approx 10^6$ sec^{-1} near the critical point). Therefore, in a first approximation one can consider the process of collision of jets at a constant temperature.

2. Statement of the Problem

Suppose that plane established motion of an incompressible liquid occurs in the region shown in Fig. 1. Two jets with the same thickness h move from infinity with the same velocity U toward each other at an angle 2γ. Near the point R of intersection of the asymptotes of the outer boundaries of the oncoming jets they separate into two diverging jets moving in opposite directions. At infinity the velocities and thicknesses of the diverging jets are V_1, V_2, k_1, and k_2, respectively. Let the flow be symmetrical relative to the bisector of the angle 2γ and let a single point O at which the velocity equals zero exist in the flow region. By virtue of the flow symmetry, the point O lies on the bisector of the angle 2γ. We introduce Cartesian coordinates x and y so that the point O lies at the origin of coordinates and the x axis is directed along the axis of symmetry within the angle 2γ. The velocity components along the x and y axes are designated as u and v; in abbreviated notation we shall understand $x_1 = x$, $x_2 = y$, $u_1 = u$, and $u_2 = v$. The stress tensor σ_{ij} is connected with the deformation-rate tensor $\dot{\varepsilon}_{ij}$ by the relation

$$\sigma_{ij} = -P\delta_{ij} + A(T)\, I^{\alpha}\dot{\varepsilon}_{ij},\ i, j = 1, 2,\ \dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.1}$$

where $I = \sqrt{\dot{\varepsilon}_{11}^2 + \dot{\varepsilon}_{12}^2}$; $A(T) = \frac{\sigma^0{}_y}{\sqrt{3}} e^{D/T}$; $\alpha = -1 + n$; $n > 0$; $P = -\frac{1}{2}\sigma_{ii}$; T is the temperature; n, $\sigma^{\circ}{}_y$, and D are constants of the material. The problem was analyzed in [21] for $\alpha = 0$ and T = const. There is no surface tension. We neglect gravitational forces in comparison with frictional and inertial forces. Let the temperature T of the medium be constant.

Eliminating σ_{ij} from the equations of motion of the medium under stress using (2.1) and carrying out the normalization

$$x \to x/h,\ y \to y/h,\ u \to u/U,\ v \to v/U,\ P \to P/\rho U^2,\ I = Ih/U, \tag{2.2}$$

for T = const we can obtain

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial P}{\partial x} + \frac{I^{\alpha}}{B}\left\{\Delta u + \frac{\alpha}{I}\left(\dot{\varepsilon}_{11}\frac{\partial I}{\partial x} + \dot{\varepsilon}_{12}\frac{\partial I}{\partial y}\right)\right\}, \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial P}{\partial y} + \frac{I^{\alpha}}{B}\left\{\Delta v + \frac{\alpha}{I}\left(\dot{\varepsilon}_{12}\frac{\partial I}{\partial x} + \dot{\varepsilon}_{22}\frac{\partial I}{\partial y}\right)\right\}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \quad B = \frac{\rho U^{1-\alpha} h^{1+\alpha}}{A}. \end{aligned} \tag{2.3}$$

We set up the boundary conditions for (2.3) by requiring that the stress vector be reduced to zero at the free boundaries Σ_k (k = 1, 2, 3, 4) and that the liquid not penetrate through Σ_k. Introducing the unit vectors τ_1 and τ_2 of the outward normal and tangent to the free boundary, we write

$$\mathbf{V}|_{\Sigma}\cdot\tau_1 = 0,\ \tau_1\cdot\sigma|_{\Sigma}\cdot\tau_2 = 0,\ \tau_1\cdot\sigma|_{\Sigma}\cdot\tau_1 = 0,\ \Sigma = \Sigma_k,\ k = 1, 2, 3, 4. \tag{2.4}$$

The shape of the free boundaries is assumed to be unknown. The problem of finding the unknown functions u, v, and P and the shape of the free boundaries Σ_k (k = 1, 2, 3, 4) satisfying (2.3) and (2.4) will be called the basic problem.

It seems difficult to find the solution of the basic problem at present. Therefore, in this report we consider a simplified problem, for which we introduce an

Additional Assumption. We assume that in any finite internal subregion G_1, containing the point O, of the flow region G the functions u and v have finite third derivatives, the function I(x, y) has a lower bound, and

$$\omega = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \to 0 \quad \text{as} \quad B \to \infty. \tag{2.5}$$

The simplified problem consists in determining the unknown functions u, v, and P satisfying (2.3)-(2.5) and the shape of the free boundaries Σ_k (k = 1, 2, 3, 4). We shall solve the simplified problem approximately for sufficiently large B (B >> 1).

Discussion of the Additional Assumption. Let us differentiate the first equation of (2.3) with respect to y and the second with respect to x. Taking their difference, we obtain

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = |\mathbf{V}|\frac{d\omega}{ds} = \frac{I^{\alpha}}{B}\left\langle \Delta\omega + \frac{\alpha(\alpha-1)}{I^2}\left\{\dot{\varepsilon}_{12}\left[\left(\frac{\partial I}{\partial y}\right)^2 - \left(\frac{\partial I}{\partial x}\right)^2\right] + 2\dot{\varepsilon}_{11}\frac{\partial I}{\partial x}\frac{\partial I}{\partial y}\right\} + \frac{\alpha}{I}\left\{\dot{\varepsilon}_{12}\left[\frac{\partial^2 I}{\partial y^2} - \frac{\partial^2 I}{\partial x^2}\right] + 2\dot{\varepsilon}_{11}\frac{\partial^2 I}{\partial x \partial y}\right\}\right\rangle, \tag{2.6}$$

where $d\omega/ds$ is the derivative of the vorticity along a stream line. Since u and v have finite third derivatives by assumption and $I(x, y) \geq \kappa > 0$, the product $I^{\alpha}\langle\ldots\rangle$ in (2.6) is finite. Therefore, $d\omega/ds \to 0$ and $\omega \to \text{const}$ as $B \to \infty$ on internal stream lines not containing the point 0. Since a plane-parallel flow ($\omega = 0$) is assigned in the oncoming jets, it is plausible that $\omega \to 0$ in the region G_1 as $B \to \infty$. On the other hand, if $\omega \to 0$ as $B \to \infty$ in the region G_1, then $u, v \to u_0, v_0$ (u_0 and v_0 are harmonic functions). But u_0 and v_0 are infinitely differentiable, while $I_0 = 2\sqrt{(\partial u_0/\partial x)^2 + (\partial u_0/\partial y)^2}$ is reduced to zero only at infinity. Therefore, the additional assumption is consistent and seems plausible. The reduction of σ_{12} to zero at the free boundary means that

$$\dot{\varepsilon}_{12}|_{\Sigma} = 0, \quad \Sigma = \Sigma_k, \ k = 1, 2, 3, 4. \tag{2.7}$$

From the condition (2.7) it follows [21] that as $B \to \infty$,

$$\omega|_{\Sigma} \to -U/R \neq 0,$$

where R is the radius of curvature of the free boundary at the point under consideration. Consequently, as $B \to \infty$ a vorticity boundary layer should form near the free boundary. Thus, the additional assumption (2.5) is equivalent to the assumption of a boundary layer for sufficiently large B. Outside the boundary layer the vorticity will be taken as neglibily small, and the flow as coinciding with inviscid flow.

The velocities u_0 and v_0 of inviscid flow satisfy the equations of motion of a Newtonian liquid (the Navier–Stokes equations). Noncorrespondence arises only when one tries to satisfy the boundary conditions (2.4). This noncorrespondence is the formal basis for the introduction of a boundary layer near the free surfaces in the case of a viscous liquid. In the case of a pseudoplastic liquid u_0 and v_0 are not solutions of the system (2.3), generally speaking, for which the terms inside the brackets to α/I on the right sides of the equations are mixed.

The disagreement will be large in a region with high deformation rates. Near the critical point, however, the surface $I(x, y)$ has a maximum and the terms to α/I are reduced to zero. Below we shall attempt to describe the differences between pseudoplastic and inviscid flow near free boundaries and far from the critical point for $B \gg 1$ in the boundary-layer approximation. Therefore, we shall assume that outside the boundary layer the pseudoplastic corrections to the inviscid flow characteristics are negligibly small compared with the corrections within the boundary layer.

The equations for a boundary layer near solid walls can be obtained using an asymptotic expansion of the solution with respect to the small parameter $1/B^{l}$ [19], as well as by an order-of-magnitude estimate of the terms in the equations of motion [20]. It is hard to give preference to either of these methods from the point of view of rigor of the presentation. To derive the boundary-layer equations in our case we use the method of an order-of-magnitude estimate of terms.

3. Simplification of the Equations of Motion

According to the condition (2.5), in the limit as $B \to \infty$,

$$\partial u/\partial y - \partial v/\partial x = 0 \tag{3.1}$$

on internal stream lines. The condition (3.1) together with the third equation of (2.3) represent the Cauchy–Riemann conditions for the analytic function $u_0 - iv_0$. The solution of the problem of the collision of incompressible plane jets with the condition (3.1) is known [1, 2]; it is found by the method of conformal mappings. Since a boundary layer exists near the boundaries Σ_k for $B \gg 1$, the region of real flow approximately coincides with the region of inviscid flow. Therefore, instead of the Cartesian coordinates x and y we can choose curvilinear orthogonal coordinates φ and ψ corresponding to inviscid flow, $d(\varphi + i\psi)/d(x + iy) = u_0 - iv_0$, where i is the imaginary unit.

We mark the coordinate φ by the index 1, and ψ by the index 2. Following [22], we write the equations of motion in stresses:

$$\nabla\left(\frac{\mathbf{V}^2}{2}\right) - \boldsymbol{\omega}\times\mathbf{V} = -\nabla P + \frac{1}{H_1 H_2}\left\{\frac{\partial}{\partial\varphi}(s_{11}H_2\eta_1 + s_{12}H_2\eta_2) + \frac{\partial}{\partial\psi}(s_{21}H_1\eta_1 + s_{22}H_1\eta_2)\right\}. \tag{3.2}$$

Here V is the velocity ($V = w\eta_1 + g\eta_2$); ω is the vorticity perpendicular to the plane of flow; H_1 and H_2 are the Lamé constants; η_1 and η_2 are unit vectors along the coordinates φ and ψ. Through a geometrical analysis we can ascertain that

$$\frac{\partial \eta_1}{\partial \varphi} = -\frac{\eta_2}{H_2}\frac{\partial H_1}{\partial \psi}, \quad \frac{\partial \eta_1}{\partial \psi} = \frac{\eta_2}{H_1}\frac{\partial H_2}{\partial \varphi},$$
$$\frac{\partial \eta_2}{\partial \varphi} = \frac{\eta_1}{H_2}\frac{\partial H_1}{\partial \psi}, \quad \frac{\partial \eta_2}{\partial \psi} = -\frac{\eta_1}{H_1}\frac{\partial H_2}{\partial \varphi}. \tag{3.3}$$

Taking the values of s_{ij} from (2.1) and using (3.3), we can write Eqs. (3.2) in the form

$$\frac{w}{H_1}\frac{\partial w}{\partial \varphi} + \frac{g}{H_2}\frac{\partial w}{\partial \psi} + \frac{g}{H_1 H_2}\left(w\frac{\partial H_1}{\partial \psi} - g\frac{\partial H_2}{\partial \varphi}\right) = -\frac{1}{H_1}\frac{\partial P}{\partial \varphi} + \frac{I^\alpha}{B}\left\{\Delta w + \right. \tag{3.4}$$
$$\left. + \frac{\alpha}{I^2}\left[\frac{e_{11}^2}{H_1}\frac{\partial e_{11}}{\partial \varphi} + \frac{e_{11}e_{12}}{H_1}\frac{\partial e_{12}}{\partial \varphi} + \frac{e_{11}e_{12}}{H_2}\frac{\partial e_{11}}{\partial \psi} + \frac{e_{12}^2}{H_2}\frac{\partial e_{12}}{\partial \psi}\right]\right\},$$
$$\frac{g}{H_2}\frac{\partial g}{\partial \psi} + \frac{w}{H_1}\frac{\partial g}{\partial \varphi} \frac{w}{H_1 H_2}\left(g\frac{\partial H_2}{\partial \varphi} - w\frac{\partial H_1}{\partial \psi}\right) = -\frac{1}{H_2}\frac{\partial P}{\partial \psi} + \frac{I^\alpha}{B}\left\{\Delta g + \right.$$
$$\left. + \frac{\alpha}{I^2}\left[\frac{e_{11}e_{12}}{H_1}\frac{\partial e_{11}}{\partial \varphi} + \frac{e_{12}^2}{H_1}\frac{\partial e_{12}}{\partial \varphi} - \frac{e_{11}^2}{H_2}\frac{\partial e_{11}}{\partial \psi} - \frac{e_{11}e_{12}}{H_2}\frac{\partial e_{12}}{\partial \psi}\right]\right\},$$
$$\frac{1}{H_1 H_2}\left\{\frac{\partial w H_2}{\partial \varphi} + \frac{\partial g H_1}{\partial \psi}\right\} = 0,$$
$$e_{11} = \frac{2}{H_1 H_2}\left(H_2\frac{\partial w}{\partial \varphi} + g\frac{\partial H_1}{\partial \psi}\right) = -e_{22}, \quad e_{12} = \frac{1}{H_1 H_2}\left(\frac{H_1\partial w - w\partial H_1}{\partial \psi} + \frac{H_2\partial g - g\partial H_2}{\partial \varphi}\right), \quad I = \sqrt{e_{11}^2 + e_{12}^2}.$$

The third equation in (3.4) is the condition of incompressibility. It is easy to verify that $H_1 = H_2 = 1/w_0$, where w_0 is the velocity of inviscid flow along a stream line. By Δw and Δg we denote the expressions

$$\Delta w = \frac{1}{H_1 H_2}\left\{\frac{1}{H_1}\frac{\partial e_{11}H_1 H_2}{\partial \varphi} + \frac{1}{H_2}\frac{\partial e_{12}H_1 H_2}{\partial \psi}\right\} + \frac{e_{11}}{H_2}\frac{\partial H_2/H_1}{\partial \varphi} + \frac{e_{12}}{H_1}\frac{\partial H_1/H_2}{\partial \psi},$$
$$\Delta g = \frac{1}{H_1 H_2}\left\{\frac{1}{H_2}\frac{\partial e_{22}H_1 H_2}{\partial \psi} + \frac{1}{H_1}\frac{\partial e_{21}H_1 H_2}{\partial \varphi}\right\} + \frac{e_{22}}{H_1}\frac{\partial H_1/H_2}{\partial \psi} + \frac{e_{21}}{H_2}\frac{\partial H_2/H_1}{\partial \varphi}.$$

In Eqs. (3.4) we estimate the terms in order of magnitude within the boundary layer by analogy with what was done in [21, 23] for a Newtonian liquid. We introduce the corrections z, g, and p to the velocity w_0 and the pressure P_0 of inviscid flow: $w = w_0(1 + z)$, $P = P_0 + p$, and g is perpendicular to a stream line of inviscid flow.

We denote the thickness of the boundary layer as δ. We assume that z, g, p, $\delta \ll 1$ and outside the boundary layer z and g have a higher order of smallness. Within the boundary layer $w_0 \approx 1$ and

$$\partial w_0/\partial \varphi = O(\delta), \ \partial w_0/\partial \psi = O(1), \ \partial^2 w_0/\partial \varphi^2 = O(\delta),$$
$$\partial^2 w_0/\partial \varphi\partial \psi = O(1), \ \partial^2 w_0/\partial \psi^2 = O(1). \tag{3.5}$$

The estimates (3.5) can be obtained using the analyticity of the function $\varphi + i\psi$ of the argument $\ln w_0 - i\theta$, as well as the expressions $\partial\psi/\partial\theta$ and $\partial\psi/\partial \ln w_0$ [24]. For $\partial w_0/\partial\varphi$, and $\partial w_0/\partial\psi$ with $|1 - w_0| \ll 1$, for example, we have

$$\left(\frac{\partial w_0}{\partial \varphi}\right)_\psi = -\frac{\left(\dfrac{\partial\psi/\partial\theta}{\partial\psi/\partial \ln w_0}\right)}{\dfrac{\partial\varphi}{\partial\theta}\left[\left(\dfrac{\partial\psi/\partial\theta}{\partial\psi/\partial \ln w_0}\right)^2 + 1\right]} = (1 - w_0)\, f_1(\theta, \gamma), \tag{3.5a}$$

$$\left(\frac{\partial w_0}{\partial \psi}\right)_\varphi = \frac{1}{\dfrac{\partial\psi}{\partial \ln w_0}\left[\left(\dfrac{\partial\psi/\partial\theta}{\partial\psi/\partial \ln w_0}\right)^2 + 1\right]} = f_2(\theta, \gamma), \tag{3.5b}$$

where $|f_1| \to \pi$ and $|f_2| \to 0$ as $\varphi \to \pm\infty$ while $|f_1|$, $|f_2| < f_0(\gamma)$ for finite φ. For not very small γ the quantity f_0 has the order of several units, so that f_1, $f_2 = O(1)$, while since $(1 - w_0) \approx \delta(\partial w_0/\partial\psi)|_\Sigma$, we obtain the first two estimates of (3.5). Estimates for the second derivatives are obtained similarly.

To estimate the derivatives of the corrections, we examine the third equation in (3.4),

$$\frac{\partial(1+z)}{\partial\varphi} + \frac{\partial g/w_0}{\partial\psi} = \frac{\partial z}{\partial\varphi} + g\frac{\partial(1/w_0)}{\partial\psi} + \frac{1}{w_0}\frac{\partial g}{\partial\psi} = 0.$$

The variation of the corrections z and g along φ takes place in a distance on the order of one, and that along ψ in a distance on the order of δ, so that

$$\frac{\partial z}{\partial\varphi} + g\frac{\partial(1/w_0)}{\partial\psi} + \frac{1}{w_0}\frac{\partial g}{\partial\psi} \approx z + g + \frac{g}{\delta} = 0,$$

from which

$$g = O(z\delta),$$

i.e., g is a smaller quantity of a higher order than δ or z. Consequently, the displacement of stream lines can be neglected and we can assume that the regions of real and inviscid flow coincide in a first approximation. Arguing similarly, we estimate e_{11}, e_{12}, and their derivatives in order of magnitude:

$$e_{11} \approx \frac{\partial w_0^2}{\partial\varphi} + 2\frac{\partial z}{\partial\varphi} = O(\delta + z),\ e_{12} \approx \frac{\partial w_0^2}{\partial\psi} + \frac{\partial z}{\partial\psi} = O\left(1 + \frac{z}{\delta}\right),$$
$$\frac{\partial e_{11}}{\partial\varphi} \approx \frac{\partial^2 w_0^2}{\partial\varphi^2} + 2\frac{\partial^2 z}{\partial\varphi^2} = O(\delta + z),\ \frac{\partial e_{12}}{\partial\psi} \approx \frac{\partial^2 w_0^2(1+z)}{\partial\psi^2} = O\left(\frac{z}{\delta^2}\right). \quad (3.7)$$

The following estimates are valid for Δw and Δg:

$$\Delta w = O(z/\delta^2), \quad \Delta g = O(z/\delta). \quad (3.8)$$

With allowance for (3.5)-(3.8) we can keep only the leading terms in (3.4):

$$\frac{\partial z}{\partial\varphi} = \frac{I^\alpha}{B}\left(1 + \alpha\frac{e_{12}^2}{I^2}\right)\frac{\partial^2 z}{\partial\psi^2},\ z\frac{\partial w_0^2}{\partial\psi} = \frac{\partial p}{\partial\psi}. \quad (3.9)$$

The first boundary condition of (2.4) is satisfied identically in a first approximation, from which we get

$$e_{12}|_\Sigma \approx \left(\frac{\partial w_0^2}{\partial\psi} + \frac{\partial z}{\partial\psi}\right)\Big|_\Sigma = 0, \quad (3.10)$$

while the third gives $p|_\Sigma - \sigma_{22}|_\Sigma = 0$, or

$$p|_\Sigma = -\frac{I^\alpha}{B}e_{11}\Big|_\Sigma = -\frac{\left(2\frac{\partial z}{\partial\varphi}\right)^{1+\alpha}}{B}\Bigg|_\Sigma. \quad (3.11)$$

Thus, for sufficiently large B the approximate solution of the simplified problem (2.3)-(2.5) comes down to the search for the unknown functions z and p from Eqs. (3.9) with the boundary conditions (3.10) and (3.11) to the known region $G_0(\varphi, \psi)$ of inviscid flow in the plane of the complex potential. The region $G_0(\varphi, \psi)$ is a two-sheet band with a branching point at the critical point. Conformity of the approximate solution to the real flow should be reached for B >> 1 such that, in the vicinity of the critical point $|z(0, 0)| \ll |z(0, \psi_\Sigma)|$, i.e., the boundary layer must not enclose the critical point. Then one must divide the region $G_0(\varphi, \psi)$ cutting the sheets of $G_0(\varphi, \psi)$ along the negative φ semiaxis, and splice them together to obtain two unconnected bands corresponding to the ramming and cumulative jets. Since the simplified equations of motion are no longer elliptic, we must set up the initial condition for z. Such a condition can be chosen by going far enough upstream from the critical point and setting

$$z(\varphi_0, \psi) = 0, \quad (3.12)$$

where φ_0 is the distance from the critical point.

For the regime of the collision ($2\gamma = 45°$, h = 0.4 cm, $U = 1.0 \cdot 10^5$ cm/sec) of copper plates a calculation of the similarity parameter B yields B = 45 at T = 300°K and B = 69 at T = 1100°K, i.e., the condition B >> 1 is satisfied. By varying B in this range one can roughly allow for the temperature. In the case of $\alpha = 0$ (a Newtonian liquid) the number B coincides with the Reynolds number $Re = \rho Uh/\mu$, while Eqs. (3.9) with the conditions (3.10)-(3.12) coincide with the equations and the boundary and initial conditions obtained in [21] for the case of a small constant viscosity.

4. Numerical Calculation of the Simplified Equations of Motion and a Comparison of the Calculated Results with Experiment

An analytical investigation of the equation for z in (3.9) is complicated because of the coefficient to $\partial^2 z/\partial\psi^2$ which depends on $\partial z/\partial\varphi$ and $\partial z/\partial\psi$. Therefore, this equation was investigated numerically on a computer. The following implicit finite-difference scheme was used to calculate the correction z:

$$\frac{z_m^{n+1,l+1} - z_m^n}{\tau} = \frac{I^\alpha(l)}{B}\left[1 + \alpha\frac{e_{12}^2(l)}{I^2(l)}\right]\frac{z_{m+1}^{n+1,l+1} - 2z_m^{n+1,l+1} + z_{m-1}^{n+1,l+1}}{h^2}, \tag{4.1}$$

$$e_{12}(l),\ I(l) = e_{12},\ I\left(\frac{z_m^{n+1,l} - z_m^n}{\tau},\ \frac{z_{m+1}^{n+1,l} - z_{m-1}^{n+1,l}}{2h}\right);$$

here τ and h are the steps in φ and ψ, respectively. We set up the condition

$$\frac{z_M^{n+1,l} - z_{M-1}^{n+1,l}}{h} = -\left.\frac{\partial w_0^2}{\partial\psi}\right|_\Sigma^{n+1}, \tag{4.2}$$

at the free boundary and the condition

$$z_0^{n+1,l} = z_1^{n+1,l}. \tag{4.3}$$

on the line $\psi = 0$. The calculation started from the value φ_0 where we took

$$z_m^0 = 0, \quad m = 0, 1, 2, \ldots, M. \tag{4.4}$$

The value of φ_0 was varied from −0.5 to −8.0. The solution of (4.1)-(4.4) was found by the trial-run method. At each step in φ the calculated values of z were successively improved by substituting the new derivatives $\Delta z/\tau$ and $\Delta z/h$ into the coefficient $(I^\alpha/B)(1 + \alpha e_{12}^2/I^2)$. The number of iterations l ranged from 3 to 15. After the seventh iteration the values of z^{l+1} and z^l did not differ in the sixth significant figure. The scheme (4.1) is stable. The step τ was varied in accordance with the law $\tau = \tau_1 + |\varphi/\varphi_0| \times (\tau_0 - \tau_1)$ where τ_0 is the initial step at $\varphi = \varphi_0$, and τ_1 is the step at $\varphi = 0$. In the calculation of z in a cumulative jet $\tau_0 = 0.01$-0.02 and $\tau_1 = 0.00038$-0.0019. In the calculation in a ramming jet, $\tau_0 = 0.01$-0.08 and $\tau_1 = 0.005$-0.08. The number of layers in ψ was varied from 20 to 50.

The calculation was cut off at $\varphi_* > 0$ so that the values of $z(\varphi_*, \psi)$ over a cross section φ_* = const differed in the third significant figure. For a cumulative jet, $\varphi_* = 1.0$ for $\gamma = 22.5$, 30, and 37.5° and $\varphi_* = 8.0$ for a ramming jet. The time of calculation of one variant on an M4030 computer was about 30 min.

TABLE 1

α	−0,990			−0,970			−0,950			−0,930			−0,870		
B / 2γ, deg	45	60	75	45	60	75	45	60	75	45	60	75	45	60	75
32	0,77	0,59	0,48	0,80	0,61	0,49	0,82	0,62	0,50	0,84	0,63	0,51	0,91	0,69	0,55
64	0,41	0,31	0,23	0,43	0,32	0,24	0,44	0,32	0,25	0,45	0,33	0,25	0,49	0,35	0,27
96	0,28	0,20	0,15	0,30	0,21	0,16	0,32	0,22	0,16	0,31	0,22	0,17	0,33	0,23	0,18
128	0,24	0,15	0,10	0,23	0,16	0,12	0,22	0,16	0,12	0.22	0,16	0,12	0,25	0,18	0,13
160	0,19	0,13	0,09	0,19	0,13	0,09	0,18	0,13	0,09	0,19	0,13	0,10	0,20	0,14	0,10

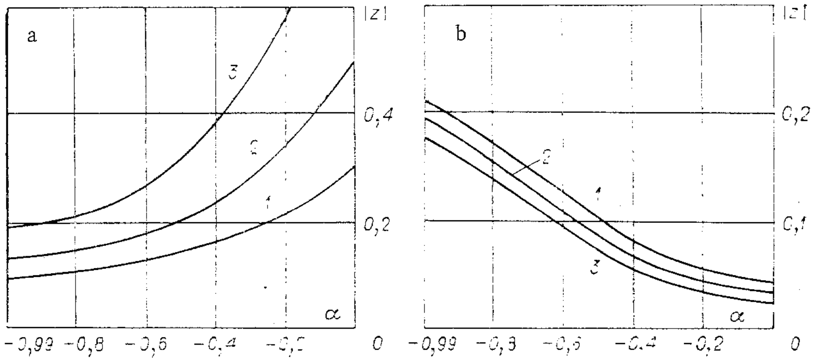

Fig. 2

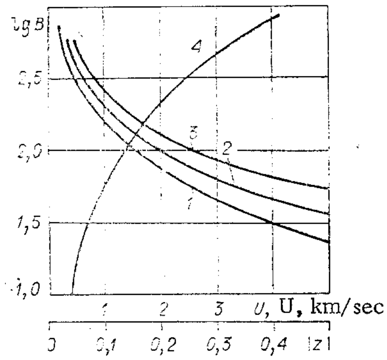

Fig. 3

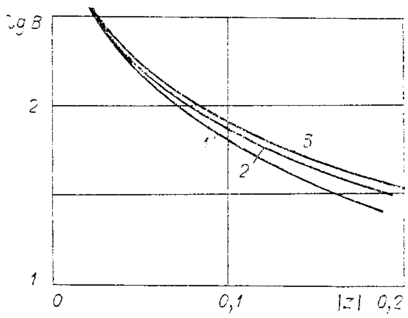

Fig. 4

The quantity z was calculated as a function of the parameters γ, B, and α. The results of calculations of the correction z in a cumulative jet are summarized in Table 1. The values of z are found at the intersection of the row with the value of B and the column with the value of 2γ in each group of three columns for a given value of α. The qualitative behavior of the function $|z(\alpha)|$ for a cumulative jet and a ram is shown in Fig. 2a and b, respectively (lines 1-3 correspond to 2γ = 75, 60, and 45° for B = 32). The dependence of $|z|$ on log B for α = −0.97 is presented in Fig. 3 (cumulative jet) and Fig. 4 (ram). The value of $|z|$ for copper can be determined approximately from the graph in Fig. 3. For this one must determine the value of log B from the value of U (curve 4) and then find the value of $|z|$ from the value of log B (lower abscissa axis) for the angles of 75, 60, and 45° from curves 1-3, respectively.

The regime of the collision (U = 1 km/sec, h = 0.4 cm) of copper plates (ρ = 8.9 g/cm^3, σ_y = 25 kg/mm^2, α = −0.97) corresponds to B = 45 at T = 300°K and B = 69 at T = 1100°K. To compare the calculated and experimental results one can choose the mean value of B = 55. The calculated correction z for 2γ = 45° and B = 55 is z = −0.52. In [4] the experimental value z ≈ −0.40 was obtained for this collision regime, i.e., the agreement between the calculated and experimental values of z is not bad.

We also calculated the shape of the indicator line by the method of [25]. The results of the calculation for B = 55, 2γ = 45°, and α = −0.97 (copper, U = 1 km/sec, h = 0.4 cm) in a comparison with the experimental data in the same collision regime are presented in Fig. 5 (the figures next to the symbols denote the multiplicity of the repetition). The agreement of the calculated and experimental displacements of the line is good. One can see that the experimental characteristics of flow both in a jet and in plates correspond fairly well to the same value of B = 55, whereas a Newtonian liquid describes these characteristics for considerably different Reynolds numbers (Re° ≈ 350 and Re_* ≈ 25, respectively). In contrast to the Reynolds number calculated in the treatment of tests on the collision of plates, to calculate B it is sufficient to know the objective characteristics ρ, σ_y, and α of the material, which can be obtained independently of tests on the collision of plates. The estimated value of B can be found from the formula $B \approx \sqrt{3}\rho U^2/\sigma_y$.

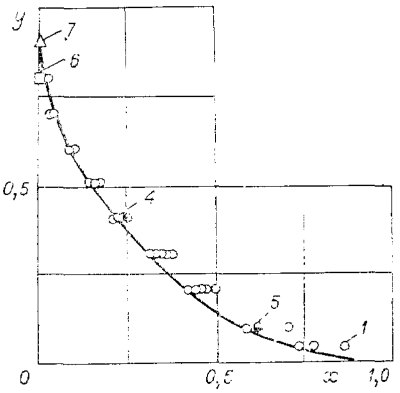

Fig. 5

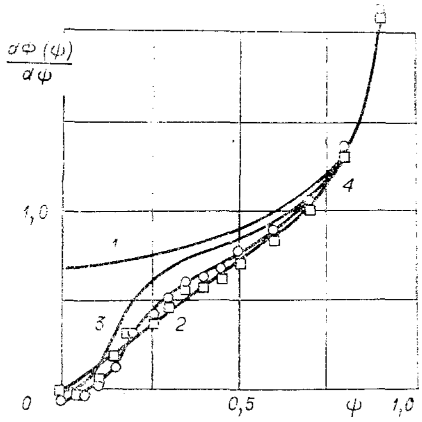

Fig. 6

It is also interesting to compare the calculated and experimental derivatives $\delta(\psi) = d\Phi(\psi)/d\psi$ of the established shape of the indicator line. Acocrding to the model of a viscous liquid, the derivative $\delta(\psi)$ grows linearly from the free boundary (Fig. 6, curve 2, Re = 70). The value of Re = 70 is taken from a comparison of the calculated shape of the line with an experiment on the collision of copper plates for U = 2.0 km/sec, h = 0.4 cm, and $2\gamma = 45°$ [25]. In this collision regime B = 170 at T = 300°K and B = 270 at T = 1100°K, and the mean value is B = 220. The function $\delta(\psi) = d\Phi(\psi)/d\psi$ calculated from the model of a power-law pseudoplastic liquid for the mean value of B has an inflection point near the free boundary (curve 3, $\alpha = -0.97$, B = 220).

We measured the experimental slope of the markers in a test with $2\gamma = 45°$, U = 2.0 km/sec, and h = 0.4 cm. The measurement was made on a microscope; the slope angle was measured to within 20 min and then the derivative was calculated as the tangent of the slope angle. The results of the measurement are presented in Fig. 6, where curve 4 approximates the experimental values of the derivative and curve 1 is the slope for an ideal liquid. On the experimental function $\delta(\psi)$ one can see an inflection point (characteristic for a pseudoplastic liquid) which is not on the calculated $\delta(\psi)$ curve based on the model of a viscous liquid. Fair numerical agreement between the experimental and calculated values of $\delta(\psi)$ is reached at B = 170 (rather than at the mean value B = 220), which probably corresponds to the well-known experimental fact of slight heating of the plates ($\Delta T \approx$ 50-100°C) in this collision regime.

It should be noted that better agreement between the calculated and experimental characteristics $|z|$ in a cumulative jet in the collision regime ($2\gamma = 45°$, copper, U = 1 km/sec, h = 0.4 cm) is reached at B = 64 (rather than at the mean value B = 55), which corresponds to heating by 700°C. In this case, the calculated value is $|z| = 0.41$ (see Table 1), while the experimental value is $|z| \approx 0.40$, as already mentioned. Although, however, such good agreement when using a very approximate flow model, which the boundary-layer approximation is for finite values of B, and with the known experimental error may also be accidental.

Thus, the model of a power-law pseudoplastic liquid, based on the Vitman–Zlatin model of high-speed deformation of metals, allows one to avoid the difficulties inherent to a viscous liquid, connected with the different calculating parameters of similarity in thin and thick jets when describing experimental facts. The assumption that there is a boundary layer near the free surface allows one to simplify the equations of motion and achieve fair quantitative agreement between the calculated and experimental characteristics of the process of plane cumulation for copper in the stage of the formation of a cumulative jet.

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